The maximum capture problem with flexible substitution patterns

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We consider the maximum capture problem with random utilities. The basic assumption is that a firm wants to locate a given number of facilities in a competitive market where customers choose the facility that maximizes their utility. Utility is treated as random. In the location science literature, so far, the corresponding choice probabilities of the customers are given by the multinomial logit model (MNL). There exist several exact mixed integer linear reformulations to the original NP–hard, non–linear program. Unfortunately, the MNL exhibits the independence from irrelevant alternatives property, i. e. constant substitution between facility locations. In contrast, the so–called mixed multinomial logit model (MXL) allows for flexible substitution patterns. Moreover, the MXL is able to approximate any random utility model arbitrarily close. In this paper, we present an intelligible mixed integer linear program for the maximum capture problem with customer demand modeled by the MXL. Empirical and managerial insights are discussed based on a unique real world case study that shows the applicability of our approach.

1 Introduction

We address the problem a firm faces when it modifies its network of facilities in a geographical market (that is, to consolidate or to expand), when there are one or more competing firms operating in the same geographical area. We assume the competitors do not react to the modification of the network of facilities of the considered firm (Müller 2013 2014). We only consider discrete, locational decisions, i. e. no decisions about the attributes of the facilities (price, for example) are made. All facilities in the market, those of the considered firm and those of possible competitors, compete for customer demand with each other. All firms want to capture as much demand as possible. A customer perceives a specific utility for each facility location. We further assume a customer chooses to patronize the facility location that maximizes his utility. Utility is treated as a random quantity, because the firm does not obtain entire information about the customers' utility function, i. e. the firm does not observe all factors that influence costumer choices. Several authors have examined this problem or closely related problems in the past (see de Palma et al. 1989, Eiselt et al. 1993, Drezner & Drezner 1996, Benati 1999, and Ndiaye 2009, for example). Benati & Hansen (2002) make restrictive assumptions about customers utility — i. e. utility is independent and identically extreme value distributed (iid EV) — such that the customers' choice probabilities for each facility location are given by the multinomial logit model (MNL). They present a problem formulation and algorithms for the so–called maximum capture problem with random utilities (MCPRU). This problem is known to be NP–hard (Benati 1999 and Benati & Hansen 2002). For the original non–linear mixed integer program (MIP) there exist three exact linear MIP reformulations so far. Benati & Hansen (2002) where the first who presented a linear MIP reformulation to the MCPRU. Their approach is based on variable substitution. Haase (2009) has proposed to employ the constant substitution pattern of the MNL in order to enable a linear MIP formulation (see also Aros-Vera et al. 2013). Finally, Zhang et al. (2012) introduced an alternative approach based on variable substitution. A comparison of the three approaches can be found in Haase & Müller (2014).

The MNL has been increasingly employed to model probabilistic customer behavior in facility location models (see Marianov et al. 2008, Lüer-Villagra & Marianov 2013, Haase & Müller 2015, and Müller & Haase 2014, for example). Moreover, we find the use of the MNL in other fields or applications of operations research, like assortment optimization (e. g. Kök & Fisher 2007 and Rusmevichientong et al. 2010), revenue management (e. g. Talluri & Van Ryzin 2004 and Suh & Aydin 2011), and public transport line planning (Klier & Haase 2015), for example. A major shortcoming of the MNL (and other related spatial interaction models)[1] in practical applications has been rarely discussed so far in the operations research literature in general and in the location science literature in particular (see McFadden 1989): the independence from irrelevant alternatives property, in short IIA (Ray 1973). Roughly speaking, the IIA yields that for a given facility location every other facility location is an equal substitute (constant substitution pattern, see Train 2009, p. 49). It is empirically well evidenced that the IIA is unlikely to hold in many spatial choice situations (see Haynes et al. 1988, Haynes & Fotheringham 1990, Anderson et al. 1992, Hunt et al. 2004, Sener et al. 2011, and Müller et al. 2012, for example). As a consequence, the predictive outcome — the MNL choice probability — is biased (see Currim 1982 and Brownstone & Train 1999, for example). The market shares based on the MNL choice probabilities are expected to be biased as well (Müller & Haase 2014). Therefore, MNL choice probabilities in facility location models are likely to produce solutions which are not optimal, because the customers supposably make locational choices different from those predicted by the MNL.

In numerous empirical studies the mixed multinomial logit model (MXL) has been applied in order to overcome the issues related to the MNL — in particular the IIA (see, for example, McFadden 1986, Train 1998, Bhat & Guo 2004, Hess & Polak 2005, Smith 2005, Briesch et al. 2013). The MXL is known to yield better predictions of the true customer behavior (i. e., choices) compared to the MNL (see, for example, Hunt et al. 2004, Allenby et al. 2005, and Jank & Kannan 2005). Moreover, the MXL is a very general choice model, because it is able to approximate any random utility model arbitrarily close (McFadden & Train 2000). The nice properties of the MXL have lead to an extensive use of this model in empirical research on (customer) choice behavior (Ben-Akiva et al. 2002, Hensher & Greene 2003). In contrast, we find only a few specific, but approximate, approaches to the MXL in the

[1] Mostly, gravity models like the Huff-Model or the multiplicative interaction model, for example. For more details see Anas (1983).

facility location planning literature so far (see Haase & Müller 2013 and Müller et al. 2009, for example). The flexibility of the MXL in terms of customer choice behavior comes at the cost of non–closed–form choice probabilities (in contrast to closed–form MNL choice probabilities). Therefore, Monte Carlo simulation methods are used to compute the MXL choice probabilities.
In this contribution we first introduce a stochastic, non–linear MIP to account for MXL choice probabilities in the maximum capture problem. Then, we propose two original linear MIP formulations based on Monte Carlo method (Niederreiter 1992, Kleywegt et al. 2002) as the corresponding deterministic equivalents (see Owen & Daskin 1998 and Laporte et al. 1994, for example). This is followed by a discussion of the MXL in Section 2. In Section 3 we present a new formulation of the maximum capture problem with random utilities based on MXL choice probabilities.

2 Customer Patronage

Consider a market where customers are located in zones denoted by nodes $\mathcal{I}$ (demand nodes). Potential (and existing) facilities of the firm and facilities of competitors are located in nodes $\mathcal{M}$. $\mathcal{M}$ might contain an artificial facility denoting a so–called "no–choice" alternative, indicating that customers might patronize no facility at all. The problem of the firm is to select r facility locations from all potential locations $\mathcal{J} \subset \mathcal{M}$ such that the total expected patronage of the firm is maximized. In the following sections we describe models and procedures to determine patronage, i. e. customer choice probabilities. We therefore rely on Train (2009) if not stated otherwise.

2.1 MNL and MXL Choice Probabilities

We assume the customers located in $i \in \mathcal{I}$ to be homogeneous in their observable characteristics like age, income and so on (Aros-Vera et al. 2013). An individual's utility for an alternative is a result of the alternative's attributes as well as the individual's characteristics. Because there are aspects of utility that the analyst (the firm) does not observe, the total utility U_{ij} of customers located in $i \in \mathcal{I}$ patronizing a facility located at $j \in \mathcal{M}$ is decomposed into a deterministic component v_{ij} and a stochastic component ϵ_{ij}:

$$U_{ij} = v_{ij} + \epsilon_{ij}. \tag{1}$$

Everything that is not included in v_{ij} (i. e., not observed) is captured by ϵ_{ij}. According to utility maximization, a customer located in $i \in \mathcal{I}$ chooses to patronize a facility located in $j \in \mathcal{M}$, iff

$$U_{ij} > U_{im} \qquad \forall\, m \in \mathcal{M}, m \neq j. \tag{2}$$

Since we do not know ϵ_{ij}, U_{ij} is a random variable. Therefore, we are only able to make probabilistic statements about the choice problem (2). We define

$$p_{ij} = \Pr\left(U_{ij} > U_{im} \quad \forall\, m \in \mathcal{M}, m \neq j\right) \tag{3}$$

as the probability that customers located in $i \in \mathcal{I}$ patronize a facility located at $j \in \mathcal{M}$. All discrete choice models (here, we consider only MNL and MXL) can be derived from 3 (McFadden 2001). If we assume that the stochastic component ϵ_{ij} is iid EV, the probability (3) is given by the MNL and

$$p_{ij}^{\text{MNL}} = \frac{e^{v_{ij}}}{\sum_{m \in \mathcal{M}} e^{v_{im}}}. \tag{4}$$

The MNL exhibits the IIA property, i. e., constant substitution patterns: the fraction $p_{ij}^{\text{MNL}}/p_{ik}^{\text{MNL}}$ remains constant whether or not a third facility is located at $m \in \mathcal{M}$. This is known as the red–bus–blue–bus paradox (Ben-Akiva & Lerman 1985, pp. 51–53). The MXL overcomes this issue. Therefore, consider a set of so–called error components $\mathcal{C}$ and the parameters

h_{ijc} observable attributes related to demand point $i \in \mathcal{I}$ and facility location $j \in \mathcal{M}$ denoting the structure of substitution for error component $c \in \mathcal{C}$, and

η_c a random term related to error component $c \in \mathcal{C}$.

Now, we decompose the stochastic utility component ϵ_{ij} of (1) as

$$\epsilon_{ij} = \sum_{c \in \mathcal{C}} \eta_c h_{ijc} + \varepsilon_{ij} \tag{5}$$

with ε_{ij} being still iid EV, then the MXL choice probabilities are derived from (3) as

$$p_{ij}^{\text{MXL}} = \int_{\eta} \left(\frac{e^{v_{ij} + \sum_{c \in \mathcal{C}} \eta_c h_{ijc}}}{\sum_{m \in \mathcal{M}} e^{v_{im} + \sum_{c \in \mathcal{C}} \eta_c h_{imc}}} \right) f(\eta \,|\theta)\, d\eta. \tag{6}$$

$f(\eta\,|\theta)$ is a $|\mathcal{C}|$–dimensional density function characterized by moment parameters θ. There are no constraints in terms of the density function f. Any density function can be used. The MNL of (4) is a special case of (6) where the mixing distribution $f(\eta\,|\theta)$ is degenerate at fixed moment parameters θ. In contrast to the constant fraction of the MNL choice probabilities $p_{ij}^{\text{MNL}}/p_{ik}^{\text{MNL}}$, the fraction $p_{ij}^{\text{MXL}}/p_{ik}^{\text{MXL}}$ depends on the existence of facility locations $m \in \mathcal{M}$ other than j and k, because the denominator in (6) does not cancel out.
Usually, the substitution pattern between facility locations is imposed by a nesting structure. Therefore, h_{ijc} might be operationalized as incidence parameters ("dummies"). For a given $i \in \mathcal{I}$ and $c \in \mathcal{C}$, h_{ijc} equals 1 for all $j \in \mathcal{J}$ that belong to the same nest, i. e. those facility locations that are close substitutes to each other. To make this more visible consider $\mathcal{M} = \mathcal{J} = \{A, B, C\}$ and $\mathcal{C} = \{1, 2\}$. Let us assume A shares unobserved attributes with B and B shares unobserved attributes with C, but A and C do not share any unobserved attributes. Furthermore, the substitution pattern is the same for all $i \in \mathcal{I}$. Then we would specify the error components as:

$$\begin{aligned}
\eta_1 \cdot h_{iA1} + \eta_2 \cdot h_{iA2} &= \eta_1 \cdot 1 + \eta_2 \cdot 0 \\
\eta_1 \cdot h_{iB1} + \eta_2 \cdot h_{iB2} &= \eta_1 \cdot 1 + \eta_2 \cdot 1 \\
\eta_1 \cdot h_{iC1} + \eta_2 \cdot h_{iC2} &= \eta_1 \cdot 0 + \eta_2 \cdot 1
\end{aligned}$$

The degree of substitution is determined by η_c and $f(\eta\,|\theta\,)$, respectively.

2.2 Simulation Procedures to Determine MXL Choice Probabilities

The MXL allows for great flexibility concerning the substitution patterns between facility locations. However, this flexibility comes at the cost of a non–closed formulation of the MXL choice probabilities p_{ij}^{MXL}. Fortunately, they can be easily simulated: The MXL probabilities p_{ij}^{MXL} of (6) are a weighted average of the MNL of (4), evaluated at different values of the $|\mathcal{C}|$–dimensional vector η, with the weight given by the density of $f(\eta\,|\theta\,)$. Therefore, we rewrite (6) as

$$p_{ij}^{\text{MXL}} = \int_{\eta} \pi_{ij}(\eta)\, f(\eta\,|\theta\,)\, d\eta, \tag{7}$$

where

$$\pi_{ij}(\eta) = \frac{e^{v_{ij}+\sum_{c\in\mathcal{C}}\eta_c h_{ijc}}}{\sum_{m\in\mathcal{M}} e^{v_{im}+\sum_{c\in\mathcal{C}}\eta_c h_{imc}}}. \tag{8}$$

Note, for a given η (8) are MNL choice probabilities as given by (4). p_{ij}^{MXL} is approximated through simulation for given θ by

Procedure A1

For each $i \in \mathcal{I}$

1. draw a realization of η from $f(\eta\,|\theta\,)$ and label it η^s, with the superscript $s = 1$ referring to the first draw. By this, we get values for $\eta_c^s \;\forall\; c \in \mathcal{C}$.
2. Compute $\pi_{ij}(\eta^s)$ of (8) for this draw.
3. Repeat steps 1 and 2 S times with $s = 1, \ldots, S$ and average the results:

$$\check{p}_{ij}^{\text{MXL}} = \frac{1}{S}\sum_{s=1}^{S} \pi_{ij}(\eta^s). \tag{9}$$

The outcome of this simulation procedure, the simulated probability $\check{p}_{ij}^{\text{MXL}}$, is an unbiased estimator of p_{ij}^{MXL} by construction. Its variance decreases as S increases, and $\sum_{j\in\mathcal{M}} \check{p}_{ij}^{\text{MXL}} = 1 \;\forall\; i \in \mathcal{I}$.
Within a second procedure we immediately exploit (3) by applying a so–called accept-reject-simulator to approximate the MXL choice probabilities of (6) as follows:

Procedure A2

For each $i \in \mathcal{I}$

1. draw a realization of η from $f(\eta\,|\theta\,)$ and label it η^s, with the superscript $s = 1$ referring to the first draw.

2. For each $j \in \mathcal{M}$ draw a realization of ε_{ij} from the extreme value distribution and label it ε_{ij}^s with the superscript $s = 1$ referring to the first draw.

3. Now compute the total utility for all $j \in \mathcal{M}$ as

$$U_{ij}^s = v_{ij} + \sum_{c \in \mathcal{C}} \eta_c^s h_{ijc} + \varepsilon_{ij}^s \tag{10}$$

4. Compute

$$a_{ijs} = \begin{cases} 1, & \text{if } U_{ij}^s > U_{ik}^s \quad \forall\, k \in \mathcal{M}, k \neq j \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

   That is, a_{ijs} equals one, if j is the facility location that maximizes the utility of customers located in i given draw s ($a_{ijs} = 1$ is called an accept).

5. Repeat steps 1 to 4 S times with $s = 1, \dots, S$. Then, the simulated MXL choice probability is the proportion of draws that are accepts:

$$\check{p}_{ij}^{\text{MXL}} = \frac{1}{S} \sum_{s=1}^{S} a_{ijs}. \tag{12}$$

Again, $\check{p}_{ij}^{\text{MXL}}$ is an unbiased estimator of p_{ij}^{MXL} by construction. The variance of $\check{p}_{ij}^{\text{MXL}}$ decreases as S increases, and $\sum_{j \in \mathcal{M}} \check{p}_{ij}^{\text{MXL}} = 1 \; \forall\, i \in \mathcal{I}$. (9) and (12) yield different deterministic equivalents to the maximum capture problem with flexible substitution patterns as shown in the next section.

3 Maximum Capture Problem with Flexible Substitution Patterns MCPFS

3.1 Model Formulations

Concerning the problem statement of the beginning of Section 2, we additionally define the locational decision variable Y_j, attaining the value 1 if a (new) facility is located at $j \in \mathcal{J}$ and 0 otherwise. Let q_i be the number of customers located in $i \in \mathcal{I}$. Using the MXL choice probabilities given in (6), the maximum capture problem with flexible substitution patterns (MCPFS) can be formulated as

Program P1

$$\text{Maximize} \quad F^{\text{P1}} = \sum_{i \in \mathcal{I}} q_i \sum_{j \in \mathcal{J}} \int_{\eta} \left(\frac{Y_j e^{v_{ij} + \sum_{c \in \mathcal{C}} \eta_c h_{ijc}}}{\sum_{m \in \mathcal{M}} Y_m e^{v_{im} + \sum_{c \in \mathcal{C}} \eta_c h_{imc}}} \right) f\left(\eta \,|\theta\right) d\eta \tag{13}$$

subject to

$$\sum_{j \in \mathcal{J}} Y_j = r \tag{14}$$

$$Y_j \in \{0,1\} \qquad \forall\, j \in \mathcal{J} \tag{15}$$

The objective function (13) maximizes the expected patronage measured in numbers of clients. (14) ensures the establishment of r facilities. Note, if $\mathcal{C} = \emptyset$ (and $S = 1$), then **P1** becomes the MCPRU as proposed by Benati & Hansen (2002), denoted as $\mathbf{P1}^{\text{MNL}}$. There exists no analytical solution to the stochastic, non–linear, MIP **P1**. However, we can reformulate **P1** as a linear MIP using (9) instead of (6) and a linear reformulation of (8) as proposed by Haase (2009). The former suggestion yields the deterministic equivalent to **P1** by sample average approximation (Kleywegt et al. 2002, Birge & Louveaux 1997). The latter yields one a tight linear MIP formulation of the deterministic equivalent (Haase & Müller 2014). One might formulate **P1** as a two-stage facility location problem as outlined in Snyder (2006) and Klein Haneveld & van der Vlerk (1999). However, we remain with the compact formulation to make the coherences with the MCPRU used in Benati & Hansen (2002), Aros-Vera et al. (2013), and Haase & Müller (2014) more visible. We additionally define the parameter

$$p_{ijs} = \frac{e^{v_{ij} + \sum_{c \in \mathcal{C}} \eta_c^s h_{ijc}}}{\sum_{k \in \mathcal{M} \setminus \mathcal{J}} e^{v_{ik} + \sum_{c \in \mathcal{C}} \eta_c^s h_{ikc}} + e^{v_{ij} + \sum_{c \in \mathcal{C}} \eta_c^s h_{ijc}}},$$

which is the choice probability of clients in i for patronizing a firm's facility located at j for draw s given that j is the only own facility established. I. e., the choice set consists of the one new facility and all competitors' facilities. p_{ijs} is the probability of the firm's patronage, whereas $(1 - p_{ijs})$ is the competitors' patronage. Thus, the latter term is the "no–choice" alternative from the firm's point of view (the cumulated choice probability for patronizing competitors' facilities or patronizing no facility at all).
The non–negative variables X_{ijs} represent the probability (i. e., the fraction) of customers located in $i \in \mathcal{I}$ patronizing a facility located at $j \in \mathcal{J}$ for draw s, and $\tilde{X}_{is}$ denoting the cumulative choice probabilities for the competitors' facilities for demand point $i \in \mathcal{I}$ and draw s. The linear deterministic equivalent to **P1** is given by MIP

Program P2

$$\text{Maximize} \quad F^{\text{P2}} = \sum_{i \in \mathcal{I}} q_i \sum_{j \in \mathcal{J}} \frac{1}{S} \sum_{s=1}^{S} X_{ijs} \tag{16}$$

subject to (14), (15), and

$$\tilde{X}_{is} + \sum_{j \in \mathcal{J}} X_{ijs} \leq 1 \qquad \forall\, i \in \mathcal{I}; s = 1, \ldots, S \tag{17}$$

$$X_{ijs} \leq p_{ijs} Y_j \qquad \forall\, i \in \mathcal{I}; j \in \mathcal{J}; s = 1, \ldots, S \tag{18}$$

$$X_{ijs} \leq \frac{p_{ijs}}{1 - p_{ijs}} \tilde{X}_{is} \qquad \forall\, i \in \mathcal{I}; j \in \mathcal{J}; s = 1, \ldots, S \tag{19}$$

$$X_{ijs} \geq 0 \qquad \forall\, i \in \mathcal{I}; j \in \mathcal{J}; s = 1, \ldots, S \tag{20}$$

$$\tilde{X}_{is} \geq 0 \qquad \forall\, i \in \mathcal{I}; s = 1, \ldots, S \tag{21}$$

The objective function (16) maximizes the simulated expected patronage measured in numbers of clients. Analogous to Haase & Müller (2015, p. 277), (17)–(19) together with the objective function are a linear reformulation of the choice probabilities in (13). (17) ensure that a demand node i's final choice probabilities for going for the firm's facilities as well patronizing the competitors' facilities sum up to 1. The linking constraints (18) allow choice probabilities for a facility to be greater than 0 only if the corresponding facility is actually established. Allowing for p_{ijs} yields a smaller upper bound by the corresponding LP–relaxation than just using $X_{ijs} \leq Y_j$ and tighter bounds for X_{ijs} (Haase & Müller 2015), because p_{ijs} is distinctly smaller than 1. (19) ensure that the pre–calculated constant substitution ratios between the choice probabilities for any two alternatives are obeyed. They are derived from $\frac{X_{ijs}}{\tilde{X}_{is}} = \frac{p_{ijs}}{1-p_{ijs}}$. But, $X_{ijs} \neq p_{ijs}$ and $\tilde{X}_{is} \neq (1 - p_{ijs})$ (unless j is the only established facility). Since $\sum_{s=1}^{S} X_{ijs}/S$ are the approximate MXL choice probabilities of (9), $F^{\mathrm{P2}*} \simeq F^{\mathrm{P1}*}$. In particular, if $S \to \infty$, then the optimal objective function value of **P2** $F^{\mathrm{P2}*}$ converges to the optimal objective function value of **P1** $F^{\mathrm{P1}*}$ at most at the rate of $\mathcal{O}(1/\sqrt{S})$ (Shapiro 1996). The procedure **A1** that yields (9) is a so–called external sampling method (Mak et al. 1999), because sampling is performed external to (prior to) the solution procedure to solve **P2**. Note, if $\mathcal{C} = \emptyset$ and $S = 1$, then X_{ij1} are the MNL choice probabilities of (4) and **P2** becomes the linear reformulation of the MCPRU as proposed by Haase & Müller (2014), denoted as $\mathbf{P2}^{\mathrm{MNL}}$. Several authors stress that the number of draws S might be very large to achieve a "good" approximation (Verweij et al. 2003), Beraldi et al. 2004, Kall & Stein 1994, Ch. 1, Linderoth et al. 2006, Mak et al. 1999). For large S, we expect that **P2** is difficult to solve by standard IP solvers (Müller & Haase 2014). To reduce this difficulty, we consider the external sampling method **A2** and use (12) instead of (6) in a second — simpler — deterministic equivalent to **P1**. Therefore, we redefine (11) as

$$a_{ijs} = \begin{cases} 1, & \text{if } U_{ij}^{s} > U_{ik}^{s} \quad \forall\, k \in \mathcal{M} \setminus \mathcal{J}, k \neq j \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

That is, $a_{ijs} = 1$, if the utility of customers located in $i \in \mathcal{I}$ patronizing a firm's facility located at $j \in \mathcal{J}$ is larger than the utility values of each facility location of the competitor(s) $k \in \mathcal{M} \setminus \mathcal{J}$. Further, we consider the non–negative variable Z_{is} that — given draw s — equals one if customers located in $i \in \mathcal{I}$ choose to patronize a located facility of the firm (0, otherwise). Then,

Program P3

$$\text{Maximize} \quad F^{\text{P3}} = \sum_{i\in\mathcal{I}} \frac{q_i}{S} \sum_{s=1}^{S} Z_{is} \tag{23}$$

subject to (14), (15), and

$$Z_{is} \leq \sum_{j\in\mathcal{J}} a_{ijs} Y_j \qquad \forall\ i \in \mathcal{I}; s = 1, \ldots, S \tag{24}$$

$$Z_{is} \in [0, 1] \qquad \forall\ i \in \mathcal{I}; s = 1, \ldots, S \tag{25}$$

is a deterministic equivalent to **P1**. For MIP **P3**, what is basically a simple exercise of ReVelle's MAXCAP (ReVelle 1986), the the same properties as for **P2** hold. As such, we assume $F^{\text{P3}*} \approxeq F^{\text{P2}*}$ for "large" S. If $\mathcal{C} = \emptyset$, then the solution of **P3** approximates the solution of MCPRU. That is, **P3** with $\mathcal{C} = \emptyset$, denoted as $\mathbf{P3}^{\text{MNL}}$, employs MNL choice probabilities.
P3 reduces the number of constraints by at most $2 \cdot |\mathcal{I}| \cdot |\mathcal{J}| \cdot S$ compared to **P2**. However, since **P3** is based on the external, crude frequency simulation procedure **A2** we expect that the number of draws S for **P3** is larger than for **P2** to obtain similar results (Lerman & Manski 1981). This is particularly true, if the choice probabilities are rather low or high, because the expected number of draws for an accept (i. e., $a_{ijs} = 1$ in (22)) is $1/\sum_{j\in\mathcal{J}} p_{ij}$. Note that **P2** can also be solved like in Mai & Lodi (2017).

3.2 Evaluation of Solutions

3.2.1 Lower Bound

Let

$$\mathcal{J}^* \left(F^{\text{P}\#*}\right) = \left\{j \in \mathcal{J} \,\middle|\, Y_j^* = 1\right\} \tag{26}$$

be an optimal solution of a given problem **P#** and $\mathcal{M}^*$ is the corresponding set of located facilities (established facility locations of the firm and the competitors). Consider this set $\mathcal{M}^*$ in (8) such that

$$\pi_{ij}^* (\eta) = \frac{e^{v_{ij} + \sum_{c\in\mathcal{C}} \eta_c h_{ijc}}}{\sum_{m\in\mathcal{M}^*} e^{v_{im} + \sum_{c\in\mathcal{C}} \eta_c h_{imc}}}. \tag{27}$$

Now, increase the number of draws in step 1 of procedure **A1** from S to S' with $S' \gg S$. Replace $\pi_{ij}(\eta^s)$ by (27) in step 2 of **A1**. Then, compute the corresponding choice probabilities in (9). Finally,

$$F_{\text{eval}}^{\text{P}\#} = \sum_{i\in\mathcal{I}} q_i \sum_{j\in\mathcal{J}^*\left(F^{\text{P}\#}\right)} \check{p}_{ij}^{\text{MXL}} \tag{28}$$

is the evaluated objective function value for problem $\mathbf{P\#}$. If $F^{\mathrm{P}\#} \approx F^{\mathrm{P}\#}_{\mathrm{eval}}$ then S might be sufficiently large.

3.2.2 Solution Quality

Since $F^{\mathrm{P2}*}$ and $F^{\mathrm{P3}*}$ are estimates of $F^{\mathrm{P1}*}$ we are interested in the quality of these estimates. One measure of quality reported in the literature is the sample variance (Shapiro & Philpott 2007). The sample variance is the variance of the sample S that is used to obtain $F^{\mathrm{P}\#*}$ (with $\# = 2, 3$):

$$\varsigma^2_S\left(\mathbf{P2}\right) = \frac{1}{S-1}\sum_{s=1}^{S}\left(F^{\mathrm{P2}*} - \sum_{i\in\mathcal{I}} q_i \sum_{j\in\mathcal{J}} X^*_{ijs}\right)^2 \tag{29}$$

$$\varsigma^2_S\left(\mathbf{P3}\right) = \frac{1}{S-1}\sum_{s=1}^{S}\left(F^{\mathrm{P3}*} - \sum_{i\in\mathcal{I}} q_i Z^*_{is}\right)^2 \tag{30}$$

The smaller the sample variance the more confident we are that $F^{\mathrm{P}\#*} \simeq F^{\mathrm{P1}*}$. Following Train (2009, p. 252) we expect $\varsigma^2_S\left(\mathbf{P2}\right)$ and $\varsigma^2_S\left(\mathbf{P3}\right)$ to decrease in S and $|\mathcal{I}|$.
However, we may end up with small sample variances due to unfortunate draws. For example, in step 1 of **A1** and **A2**, we might obtain $\eta^1 \approx \eta^2 \approx \cdots \approx \eta^S$. In such a case, (29) and (30) are less useful. We suggest to solve $\mathbf{P\#}$ N times (Bayraksan & Morton 2006, Romauch & Hartl 2005). For each $n = 1, \ldots, N$ we consider a different, independent sequence of draws $s^n = 1, \ldots, S$, yielding N different realizations $\eta^1_n, \ldots, \eta^S_n$. The solution corresponding to sequence n is denoted by $F^{\mathrm{P}\#*}_n$. Let $\overline{F^{\mathrm{P}\#*}}$ denote the average over N solutions, then the solution variance is given as

$$\varsigma^2_{S,N}\left(\mathbf{P\#}\right) = \frac{1}{N\left(N-1\right)}\sum_{n=1}^{N}\left(F^{\mathrm{P}\#*}_n - \overline{F^{\mathrm{P}\#*}}\right)^2. \tag{31}$$

The smaller $\varsigma^2_{S,N}\left(\mathbf{P\#}\right)$ the more confident we are that $\overline{F^{\mathrm{P}\#*}} \simeq F^{\mathrm{P1}*}$. We expect $\varsigma^2_{S,N}\left(\mathbf{P\#}\right)$ to decline in S and N. Further, (31) is a valid lower bound to problem $\mathbf{P\#}$.

4 Conclusion

We have seen that the proposed simulation based approach is able to approximate the maximum capture problem with random utilities arbitrarily close. By an intelligible modification of ReVelle's MAXCAP we can approximate the objective function value of the original problem (proven NP–hard) with a deviation of less than one percent in circa one minute given a (so–called large sized) problem set (400 demand points and 50 potential locations) using GAMS/CPLEX. The presented case study verifies the applicability and appropriateness of our approach. In particular, we see that a small number of draws seems to be sufficient for the simulation. A second contribution of this paper is the finding that the presented approach is general in terms of the underlying utilities. In contrast to Benati & Hansen (2002), who assume the stochastic part is iid EV, we do not make any restricting assumptions about the stochastic part of utility. Roughly speaking, we are able to approximate the market

share function (objective function) for a wide range of discrete choice models (such as nested logit, mixed logit, and probit).
Given the quality of the approximate approach, the use of sophisticated, tailored software (algorithms) to solve the maximum capture problem with random utilities becomes questionable. Using our approach, practioners are enabled to use state of the art solvers to solve their problems. Researchers are provided with a theoretically sound and capable approach to approximate the maximum capture problem with random utilities in reasonable time.
Future research may focus on variance reducing methods concerning the simulation in order to decrease computational effort while keeping the quality of the solution. We intend to consider additional constraints or a multi–period approach. Particularly, the extension of the approach to a design problem (considering decisions on opening times or capacities) seems to be interesting. Another interesting future research direction is the integration of the choice set generation process, i. e. the construction of $\mathcal{M}_i$. It would be interesting to investigate how multiple choices of facility locations can be considered.

References

Allenby, G., Fennell, G., Huber, J., Eagle, T., Gilbride, T., Horsky, D., Kim, J., Lenk, P., Johnson, R., Ofek, E., Orme, B., Otter, T., & Walker, J. (2005). Adjusting choice models to better predict market behavior. *Marketing Letters*, *16*(3-4), 197–208.

Anas, A. (1983). Discrete choice theory, information theory and the multinomial logit and gravity models. *Transportation Research - B*, *17*, 13–23.

Anderson, S., de Palma, A., & Thisse, J.-F. (1992). *Discrete Choice Theory of Product Differentiation*. MIT Press.

Aros-Vera, F., Marianov, V., & Mitchell, J. E. (2013). p-hub approach for the optimal park-and-ride facility location problem. *European Journal of Operational Research*, *226*(2), 277 – 285.

Bayraksan, G. & Morton, D. P. (2006). Assessing solution quality in stochastic programs. *Mathematical Programming B*, *108*(2-3), 495–514.

Ben-Akiva, M. & Lerman, S. (1985). *Discrete Choice Analysis, Theory and Application to Travel Demand*. Cambridge, MA: MIT Press.

Ben-Akiva, M., Mc Fadden, D., Train, K., Walker, J., Bhat, C., Bierlaire, M., Bolduc, D., Boersch-Supan, A., Brownstone, D., Bunch, D., Daly, A., De Palma, A., Gopinath, D., Karlstrom, A., & Munizaga, M. (2002). Hybrid choice models: Progress and challenges. *Marketing Letters*, *13*(3), 163–175.

Benati, S. (1999). The maximum capture problem with heterogeneous customers. *Computers & Operations Research*, *26*(14), 1351–1367.

Benati, S. & Hansen, P. (2002). The maximum capture problem with random utilities: Problem formulation and algorithms. *European Journal of Operational Research*, *143*, 518–530.

Beraldi, P., Bruni, M., & Conforti, D. (2004). Designing robust emergency medical service via stochastic programming. *European Journal of Operational Research*, *158*(1), 183–193.

Bhat, C. R. & Guo, J. (2004). A mixed spatially correlated logit model: formulation and application to residential choice modeling. *Transportation Research Part B: Methodological*, *38*(2), 147–168.

Birge, J. & Louveaux, F. (1997). *Introduction to Stochastic Programming*. Springer Verlag.

Briesch, R. A., Dillon, W. R., & Fox, E. J. (2013). Category positioning and store choice: The role of destination categories. *Marketing Science*, *32*(3), 488–509.

Brownstone, D. & Train, K. E. (1999). Forecasting new product penetration with flexible substitution patterns. *Journal of Econometrics*, *89*, 109–129.

Currim, I. S. (1982). Predictive testing of consumer choice models not subject to independence of irrelevant alternatives. *Journal of Marketing Research*, *19*(2), 208–222.

de Palma, A., Ginsburgh, V., Labbe, M., & Thisse, J.-F. (1989). Competitive location with random utilities. *Transportation Science*, *23*(4), 244–252.

Drezner, T. & Drezner, Z. (1996). Competitive facilities: market share and location with random utility. *Journal of Regional Science*, *36*, 1–15.

Eiselt, H., Laporte, G., & Thisse, J. (1993). Competitive location models: A framework and bibliography. *Transportation Science*, *27*(1), 44–54.

Haase, K. (2009). Discrete location planning. Technical Report WP-09-07, Institute for Transport and Logistics Studies, University of Sydney.

Haase, K. & Müller, S. (2013). Management of school locations allowing for free school choice. *Omega*, *41*(5), 847–855.

Haase, K. & Müller, S. (2014). A comparison of linear reformulations for multinomial logit choice probabilities in facility location models. *European Journal of Operational Research*, *232*(3), 689–691.

Haase, K. & Müller, S. (2015). Insights into clients' choice in preventive health care facility location planning. *OR Spectrum*, *37*(1), 273–291.

Haynes, K. & Fotheringham, A. (1990). The impact of space on the application of discrete choice models. *The Review of Regional Studies*, *20*, 39–49.

Haynes, K., Good, D., & Dignan, T. (1988). Discrete spatial choice and the axiom of independence from irrelevant alternatives. *Socio-Economic Planning Science*, *22*(6), 241–251.

Hensher, D. A. & Greene, W. H. (2003). The mixed logit model: The state of practice. *Transportation*, *30*(2), 133–176.

Hess, S. & Polak, J. W. (2005). Mixed logit modelling of airport choice in multi-airport regions. *Journal of Air Transport Management*, *11*(2), 59–68.

Hunt, L., Boots, B., & Kanaroglou, P. (2004). Spatial choice modelling: New opportunities to incorporate space into substitution patterns. *Progress in Human Geography*, *28*, 746–766.

Jank, W. & Kannan, P. K. (2005). Understanding geographical markets of online firms using spatial models of customer choice. *Marketing Science*, *24*(4), 623–634.

Kall, P. & Stein, W. (1994). *Stochastic Programming* (2nd ed.). John Wiley and Sons.

Klein Haneveld, W. K. & van der Vlerk, M. H. (1999). Stochastic integer programming:general models and algorithms. *Annals of Operations Research*, *85*, 39–57.

Kleywegt, A., Shapiro, A., & Homem-de Mello, T. (2002). The sample average approximation method for stochastic discrete optimization. *SIAM Journal on Optimization*, *12*(2), 479–502.

Klier, M. J. & Haase, K. (2015). Urban public transit network optimization with flexible demand. *OR Spectrum*, *37*(1), 195–215.

Kök, A. G. & Fisher, M. L. (2007). Demand estimation and assortment optimization under substitution: Methodology and application. *Operations Research*, *55*(6), 1001–1021.

Laporte, G., Louveaux, F., & van Hamme, L. (1994). Exact solution to a location problem with stochastic demands. *Transportation Science*, *28*(2), 95–103.

Lerman, S. & Manski, C. (1981). *On the use of simulated frequencies to approximate choice probabilities*, chapter 7, (pp. 305–319). MIT Press.

Linderoth, J., Shapiro, A., & Wright, S. (2006). The empirical behavior of sampling methods for stochastic programming. *Annals of Operations Research*, *142*, 215–241.

Lüer-Villagra, A. & Marianov, V. (2013). A competitive hub location and pricing problem. *European Journal of Operational Research*, *231*(3), 734–744.

Mai, T. & Lodi, A. (2017). Solving Large–Scale Competitive Facility Location Under Random Utility Maximization Models. Working Paper DS4DM-2017-007, Polytechnique Montréal.

Mak, W.-K., Morton, D. P., & Wood, R. (1999). Monte carlo bounding techniques for determining solution quality in stochastic programs. *Operations Research Letters*, *24*(1-2), 47–56.

Marianov, V., Ríos, M., & Icaza, M. J. (2008). Facility location for market capture when users rank facilities by shorter travel and waiting times. *European Journal of Operational Research*, *191*(1), 30–42.

McFadden, D. (1986). The choice theory approach to market research. *Marketing Science*, *5*, 275–297.

McFadden, D. (1989). Econometric modeling of locational behavior. *Annals of Operations Research*, *18*(1), 1–15.

McFadden, D. (2001). Economic choices. *American Economic Review*, *91*(3), 351–378.

McFadden, D. & Train, K. (2000). Mixed MNL models for discrete choice response. *Journal of Applied Econometrics*, *15*(5), 447–470.

Müller, S. (2013). Incorporating Discrete Choice Models in Competitive Facility Location Planning Models. In *20th EURO Working Group on Locational Analysis Meeting, Ankara (Turkey)*.

Müller, S. (2014). The Maximum Capture Problem with Flexible Substitution Patterns. In *ISOLDE XIII International Symposium on Locational Decisions, Naples (Italy)*.

Müller, S. & Haase, K. (2014). Customer segmentation in retail facility location planning. *Business Research*, *7*(2), 235–261.

Müller, S., Haase, K., & Kless, S. (2009). A multi-period school location planning approach with free school choice. *Environment and Planning A*, *41*(12), 2929–2945.

Müller, S., Haase, K., & Seidel, F. (2012). Exposing unobserved spatial similarity: Evidence from German school choice data. *Geographical Analysis.*, *44*, 65–86.

Ndiaye, M. (2009). Customer behaviour modelling in the maximum capture model. In *Industrial Engineering and Engineering Management, 2009. IEEM 2009. IEEE International Conference on*, (pp. 2285–2289).

Niederreiter, H. (1992). *Random Number Generation and Quasi-Monte Carlo Methods*. Society for Industrial and Applied Mathematics.

Owen, S. & Daskin, M. (1998). Strategic facility location: A review. *European Journal of Operational Research*, *111*(3), 423–447.

Ray, P. (1973). Independence of irrelevant alternatives. *Econometrica*, *41*(5), 987–991.

ReVelle, C. (1986). The maximum capture location problem, hotelling revisited on a network. *Journal of Regional Science*, *26*, 104–118.

Romauch, M. & Hartl, R. F. (2005). Dynamic Facility Location with Stochastic Demands. In O. B. Lupanov, O. M. Kasim-Zade, A. V. Chaskin, & K. Steinhöfel (Eds.), *Stochastic Algorithms: Foundations and Applications. SAGA 2005. Lecture Notes in Computer Science*, volume 3777 (pp. 180–189). Springer Berlin Heidelberg.

Rusmevichientong, P., Shen, Z.-J. M., & Shmoys, D. B. (2010). Dynamic assortment optimization with a multinomial logit choice model and capacity constraint. *Operations Research*, *58*(6), 1666–1680.

Sener, I., Pendyala, R., & Bhat, C. (2011). Accommodating spatial correlation across choice alternatives in discrete choice models: an application to modeling residential location choice behavior. *Journal of Transport Geography*, *19*(2), 294–303.

Shapiro, A. (1996). Simulation-based optimization - convergence analysis and statistical inference. *Communications in Statistics. Stochastic Models*, *12*(3), 425–454.

Shapiro, A. & Philpott, A. (2007). A tutorial on stochastic programming. online. accessed: 2017-08-09.

Smith, M. D. (2005). State dependence and heterogeneity in fishing location choice. *Journal of Environmental Economics and Management*, *50*, 319–340.

Snyder, L. (2006). Facility location under uncertainty: a review. *IIE Transactions*, *38*(7), 547–564.

Suh, M. & Aydin, G. (2011). Dynamic pricing of substitutable products with limited inventories under logit demand. *IIE Transactions*, *43*(5), 323–331.

Talluri, K. & Van Ryzin, G. (2004). Revenue management under a general discrete choice model of consumer behavior. *Management Science*, *50*(1), 15–33.

Train, K. E. (1998). Recreation demand models with taste differences over people. *Land Economics*, *74*(2), 230–239.

Train, K. E. (2009). *Discrete choice methods with simulation* (2nd ed.). Cambridge University Press.

Verweij, B., Ahmed, S., Kleywegt, A., Nemhauser, G., & Shapiro, A. (2003). The sample average approximation method applied to stochastic routing problems: a computational study. *Computational Optimization and Applications*, *24*(2), 289–333.

Zhang, Y., Berman, O., & Verter, V. (2012). The impact of client choice on preventive healthcare facility network design. *OR Spectrum*, *34*, 349–370.